M 317 assignment 4 solutions

7. Suppose a_n assumes only integer values. Under what conditions does this sequence converge?

If $\{a_n\}$ contains only integer values then the a_n are all isolated points. Then the only way for there to be a limit point for this sequence is if $a_n =$ constant for all n greater than some N.

8 Either give an example of a sequence with the following property or else state a theorem that shows why no such example is possible.

a. a sequence that is monotone increasing but is not bounded $\{a_n\} = \{n\}$

b. a sequence that converges to 6 but contains infinitely many terms that are not equal to 6 as well as infinitely many terms that are equal to 6 $a_n = 6 + \frac{1+(-1)^n}{n}$

c. an increasing sequence that is bounded but is not convergent. Not possible due to monotone seq theorem

d. a sequence that converges to 6 but no term of the sequence actually equals 6. $a_n = 6 + \frac{1}{n}$

e. a sequence that converges to 6 but contains a subsequence converging to 0. Not possible as this sequence would have two limit points, zero and six.

f. a convergent sequence with all negative terms whose limit is not negative $a_n = \frac{-1}{n}$ converges to 0 (which is not negative) but $a_n < 0$ for all n.

g. an unbounded increasing sequence containing a convergent subsequence. Not possible as the convergent subsequence would have to be bounded but any subsequence of an unbounded sequence is unbounded.

h. a convergent sequence whose terms are all irrational but whose limit is rational. $a_n = 1 + \frac{\sqrt{2}}{n}$

9 For each of the following sequences state a theorem which establishes the convergence/divergence:

a. $a_n = n^{1/3}$ not bounded, therefore divergent (theorem 2.1)

b. $a_n = \frac{n^2+3}{n+2}$ not bounded, therefore divergent (theorem 2.1)

c. $a_n = (2+10^{-n})(1+(-1)^n)$ has two limit points, 0 and 2, so it is divergent (theorem 2.3)

d. $a_n = \frac{1}{n^2+3n+2} < \frac{1}{n^2}$ so converges by squeeze theorem (theorem 2.4)

e. $a_n = 1+2^{-n}$ monotone decreasing bounded below by 1, so convergent to 1 (theorem 2.2)

f. $a_n = \sqrt{n+1}$ not bounded, therefore divergent (theorem 2.1)

g. $a_n = \sum_{k=1}^{n} \frac{1}{k}$ showed in class that $a_{2n} - a_n$ does not tend to 0 as $n \to \infty$, therefore not Cauchy and not convergent. (theorem 2.9)

h. $\{a_n\} = \left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \dots\right\}$ has two limit points, 0 and 1, so it is divergent (theorem 2.3)

14 Show that if, $a_n = \frac{n+1}{n}$ then $|a_n - a_m| \le 10^{-3}$ when $m > n > 10^3$. Is $\{a_n\}$ a Cauchy sequence?

$$
\begin{aligned}
|a_n - a_m| &= \left| \frac{n+1}{n} - \frac{m+1}{m} \right| \\
&= \left| \frac{m-n}{mn} \right| < \frac{1}{n} \quad if \quad m > n
\end{aligned}
$$

Then $|a_n - a_m| \le 10^{-3}$ when $m > n > 10^3$, hence $\{a_n\}$ is a Cauchy sequence.